3. 2-spheres.

Example 3.1. *A 2-sphere whose exterior is not simply connected.*[14]

Such a 2-sphere is the boundary X^0 of the 3-cell $p \cup \bigcup_{n=-\infty}^{\infty} f_n(U) \cup q$. It would be very simple to obtain from this a 2-sphere with both interior and exterior non-simply connected (cf. below).

Example 3.2. *A 2-sphere which is wildly imbedded even though both complementary domains are open 3-cells.*

Such a 2-sphere is the boundary Y^0 of the 3-cell

$$f_0(U_0) \cup f_0(U_+) \cup \bigcup_{n=1}^{\infty} f_n(U) \cup q.$$

The proofs in 1.2 apply with a few mild changes in the wording. This example shows that *the 3-sphere S may be decomposed into a closed 3-cell and a complementary open 3-cell in several essentially distinct ways.*

Example 3.3. *A 2-sphere whose exterior though simply connected is not an open 3-cell.*

For this example we choose Z^0, the boundary of the 3-cell

$$p \cup \bigcup_{n=-\infty}^{-1} g_n(U) \cup g_0(U_+ \cup U_0) \cup f_1(U_0 \cup U_+) \cup \bigcup_{n=2}^{\infty} f_n(U) \cup q.$$

Its exterior is homeomorphic to the complement of Z. From this example it is easy to construct a 2-sphere both of whose complementary domains are simply connected (and hence contractible) open manifolds not homeomorphic to an open 3-cell. Such a one is shown in figure 12.

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[14] Such a 2-sphere was constructed by Alexander, loc. cit., p. 11 and pp. 8–10. Our example, which is a much simpler one, has only two singular points while the Alexander examples have an infinity of singular points.

THE THEORY OF BRAIDS

By EMIL ARTIN
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THE theory of braids shows the interplay of two disciplines of pure mathematics—topology, used in the definition of braids, and the theory of groups, used in their treatment.

The fundamentals of the theory can be understood without too much technical knowledge. It originated from a much older problem in pure mathematics—the classification of knots. Much progress has been achieved in this field; but all the progress seems only to emphasize the extreme difficulty of the problem. Today we are still very far from a complete solution. In view of this fact it is advisable to study objects that are in some fashion similar to knots, yet simple enough so as to make a complete classification possible. Braids are such objects.

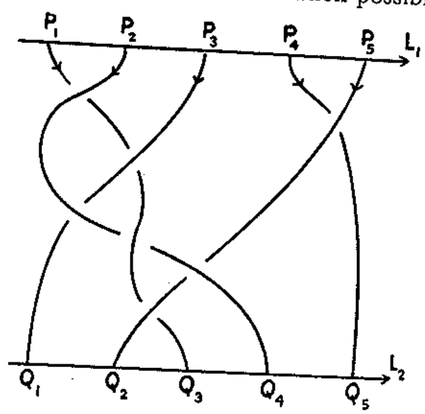

Figure 1

In order to develop the theory of braids we first explain what we call a *weaving pattern* of order n (n being an ordinary integral number which is taken to be 5 in Figure 1).

Let L_1 and L_2 be two parallel straight lines in space with given orientation in the same sense (indicated by arrows). If P is a point on L_1, Q a point on L_2, we shall sometimes join P and Q by a curve c. In our drawings we can only indicate the projection of c onto the plane containing L_1 and L_2, since c itself may be a winding curve in space.

The curves c that we shall use will be restricted in their nature by the following condition. If R is a point on the projection of c that moves from P to Q, then its distance from the line L_1 shall always increase. (Therefore a curve moving down a little, then up, and finally down again would be ruled out.) In order to have at our disposal a short name for such curves, let us call them normal curves. We orient them (by arrows) in the sense from P to Q.

Select n points on L_1. Moving along L_1 in the direction indicated by the arrow we shall call the first of the given n points P_1, the next P_2, and the last P_n. In the same way denote by $Q_1, Q_2, \ldots Q_n$, n points on the line L_2. Now we connect each point P_i with one of the points Q_j by a normal curve c_i (c_i begins at P_i and ends at some Q_j, which may or

Based on material presented in the Sigma Xi National Lectureships, 1949.

may not be Q_i). We only observe the following condition: no two of the curves c_i intersect in space. Consequently no two of the curves c_i end at the same point Q_j.

If we want to indicate this in a drawing, we have to overcome the difficulty that, although the curves do not meet in space, their projections may cross over each other. To indicate that at a certain crossing the curve c_i is below another one, we interrupt its projection slightly (this is the well-known way to indicate such occurrences in technical drawings).

The whole system of straight lines and curves shall be called a weaving pattern.

In order to explain the notion of a braid we start with a given weaving pattern and think of the lines L_1 and L_2 as being made of rigid material, whereas the curves c_i are considered as arbitrarily stretchable, contractible, and flexible. The points P_i and Q_j may also move on their lines provided their ordering is always preserved.

We subject the whole weaving pattern to an arbitrary deformation in space restricted by the following conditions:

(1) L_1 and L_2 stay parallel during the deformation (but otherwise they can be moved freely in space; their distance may change).

(2) No two of the curves c_i intersect each other during the deformation (this means that the material is "impenetrable").

(3) The curves stay normal during the deformation (but otherwise they may be stretched or contracted as the situation demands).

After such a deformation we obtain a weaving pattern that may look quite different from the one we started with. A quite tame-looking pattern may indeed (after the deformation) become hopelessly entangled.

By a braid we mean a weaving pattern together with the permission to deform it according to the previous rules. If we present a weaving pattern, it describes a braid. But infinitely many patterns will describe the same braid, namely all those that can be obtained from the given one by a deformation. The order n of the pattern shall be called the *order of the braid.*

We now have the following fundamental problem. Given two weaving patterns, is it possible to decide whether or not they describe the same braid? In other words, is it possible to decide whether or not a pattern can be deformed into a given other one?

Up to now we have considered braids of all orders n. From now on we assume n to be an arbitrary but fixed integer and restrict ourselves, without saying it explicitly, to braids of that order n.

Let now A and B be two braids. We first explain what we mean by the product AB of A and B. We select definite patterns for A and B. Call L_1, L_2, P_i, Q_i, c_i the lines, points, and curves respectively of A, and L'_1, L'_2, P'_i, Q'_i, c'_i those of B.

We deform B until the plane through L'_1 and L'_2 coincides with the plane through L_1 and L_2, and until the line L_2 coincides (including orientation) with the line L'_1, being careful to have L_1 and L'_2 on different sides of L_2. Finally we deform B until the points Q_i coincide with the points P'_i. This being achieved, we erase the line L_2, obtaining a new composed weaving pattern which shall stand as pattern for the braid AB.

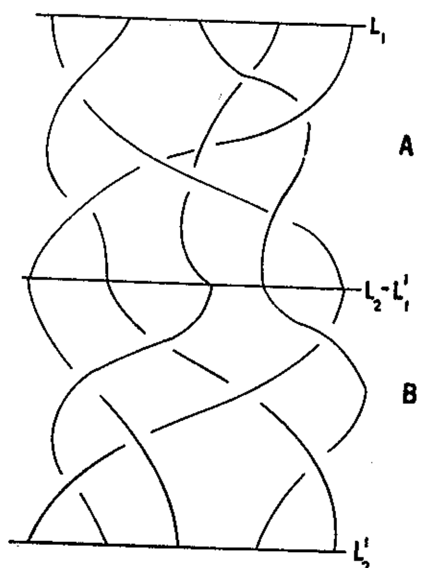

Figure 2

Intuitively speaking, this means: AB is obtained by tying the beginning of B to the end of A. Figure 2 explains the process. The reason for calling the result of this process a product lies in the fact that the process has some similarity to the ordinary multiplication of numbers. We first show:

$$(AB)C = A(BC)$$

the so-called associative law of multiplication.

What does $(AB)C$ mean? It means: form first AB and compose this with C. So tie B to A and to the result tie C. What, on the other hand, does $A(BC)$ mean? It asks us first to form BC, that is to tie C to B. The result shall be tied to A. Obviously we obtain the same pattern as $(AB)C$.

But this similarity does not go too far. For instance, the law $AB = BA$ is false in general. Very simple examples already show this. It may hold only accidentally for very special braids. In computations one must, therefore, be careful about the order of terms in a product.

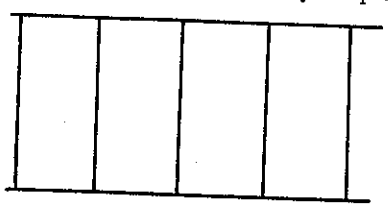

Figure 3

Let us denote by I the braid indicated in Figure 3. In its pattern the curves c_i are simply straight lines joining P_i and Q_i without crossings. If we tie I to any braid A, it is almost immediately seen that the resulting braid AI can be changed back to A; indeed the line L_2 is simply replaced by a somewhat lower line. Therefore $AI = A$ for any braid A; similarly we see $IA = A$ for any A.

Our braid I has therefore a strong resemblance to the number 1 (since $1.a = a.1$ for any number a). This explains the choice of the name I (roman one).

What does the equation $A = I$ mean? If A is originally given by some complicated pattern, then $A = I$ means that by some deformation this pattern can be changed into the pattern of Figure 3. We may say intuitively: $A = I$ means that A can be combed.

Figure 4 shows the braid A of Figure 1, and tied to it its exact reflexion on the line L_2 which we call A^{-1}. The reader can convince himself that the combined braid AA^{-1} can be disentangled if he starts removing crossings from the middle outwards. In the same way he can see that $A^{-1}A$ can be combed.

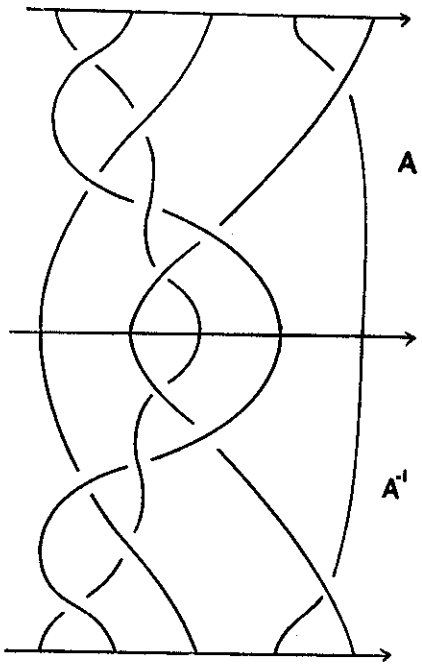

FIGURE 4

There exists therefore to any braid A another braid A^{-1} (its reflexion) such that

$$AA^{-1} = A^{-1}A = I$$

The symbol A^{-1} is chosen because of an analogy with elementary algebra where a^{-1} stands for the number $\frac{1}{a}$ so that $aa^{-1} = a^{-1}a = 1$ for any nonzero number a.

Reviewing we may say: the braids form a system of objects in which a multiplication is defined. Three properties hold for this multiplication:

(1) The associative law $(AB)C = A(BC)$ is satisfied.

(2) There is a braid called I such that $AI = IA = A$ holds for any braid A.

(3) To any braid A another braid A^{-1} can be found such that $AA^{-1} = A^{-1}A = I$.

If in these three statements we were to replace the word "braid" by the phrase "object of the system," we should obtain the exact definition of what in higher algebra one calls a "group." A group is simply a system of arbitrary objects, together with some kind of multiplication such that our three properties hold. We may say therefore: the system of all braids of order n is a group.

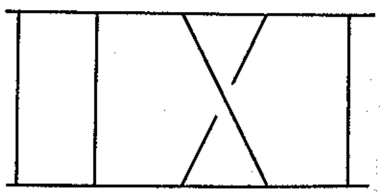

FIGURE 5

The theory of groups has been developed extensively, and its methods may be applied to our

problem. Let us look at the special braid indicated in Figure 5. Here the curve c_i goes once over the curve c_{i+1}, whereas all other curves are straight lines connecting P_j and Q_j. We shall call this braid σ_i and obtain in this fashion n–1 braids $\sigma_1, \sigma_2, \ldots \sigma_{n-1}$. ($\sigma_n$ does not exist since it would involve an n+1-st curve). The braid where c_i goes *under* c_{i+1} needs no new name. It is the reflexion of σ_i and may therefore be denoted by σ_i^{-1}.

Consider now the pattern of any braid A, for example the braid in Figure 1. In its projection two crossings may occur at exactly the same height. But it is evident that a slight deformation of braid A will produce a pattern where this does not happen.

We cut up our pattern into small horizontal sections, such that only one crossing occurs in each section. Our braid A is obtained from all these sections by tying them together again. Each of these sections is obviously either a braid σ_i or a braid σ_i^{-1} depending on the nature of the crossings. Consequently we can express A as the product of terms each of which is either a σ_i or a σ_i^{-1}.

The braid in Figure 1, for example, is given by:

$$A = \sigma_1^{-1}\,\sigma_4^{-1}\,\sigma_2^{-1}\,\sigma_1\,\sigma_2^{-1}\,\sigma_3\,\sigma_2^{-1}$$

If every element in a group can be expressed as product of some elements σ_i and their inverses, we say that the σ_i are generators of the group. We may therefore state: the $n-1$ elements σ_i are generators of the braid group.

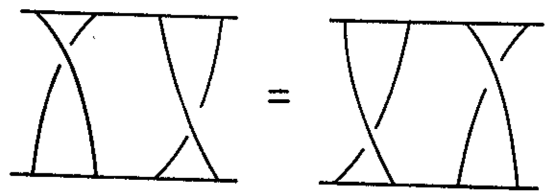

FIGURE 6

We are now in a position to describe any weaving pattern. As an example let us look at the braids in a girl's hair. A close look reveals that such a braid can be described by:

$$A = \sigma_1\,\sigma_2^{-1}\,\sigma_1\,\sigma_2^{-1}\ldots\sigma_1\,\sigma_2^{-1} = (\sigma_1\,\sigma_2^{-1})^k$$

where k is the number of times the elementary weaving pattern is repeated.

Figure 6 shows the equality $\sigma_1\,\sigma_3 = \sigma_3\,\sigma_1$. A similar figure would show $\sigma_i\,\sigma_j = \sigma_j\,\sigma_i$ if j is i+2 or more. That $\sigma_1\,\sigma_2$ is different from $\sigma_2\,\sigma_1$ can be seen by a simple sketch; in $\sigma_1\,\sigma_2$ the curve c_1 runs from P_1 to Q_3, whereas in $\sigma_2\,\sigma_1$ it runs from P_1 to Q_2.

But $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}$. Figure 7 shows it for $i = 1$; the reader readily deforms the two patterns into each other.

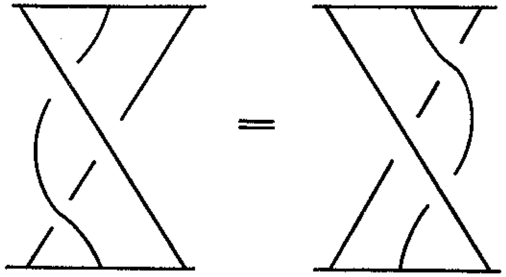

FIGURE 7

We have seen that the group has $n-1$ generators. Actually we can get away with only two: namely σ_1 and the braid

$$\alpha = \sigma_1 \sigma_2 \ldots \sigma_{n-1} \text{ (product)}$$

Let us prove the statement for $n = 5$. We have

$$\alpha \sigma_1 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \cdot \sigma_1$$

But $\sigma_4 \sigma_1 = \sigma_1 \sigma_4$; therefore $\alpha \sigma_1 = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_4$.
Now $\sigma_3 \sigma_1 = \sigma_1 \sigma_3$; hence $\alpha \sigma_1 = \sigma_1 \sigma_2 \sigma_1 \cdot \sigma_3 \sigma_4$.
From $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, we obtain $\alpha \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_4$.
Therefore $\alpha \sigma_1 = \sigma_2 \alpha$ or $\alpha \sigma_1 \alpha^{-1} = \sigma_2 \alpha \alpha^{-1} = \sigma_2 \cdot I = \sigma_2$.
Hence $\sigma_2 = \alpha \sigma_1 \alpha^{-1}$.

Similarly:

$$\alpha \sigma_2 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \cdot \sigma_2 = \sigma_1 \cdot \sigma_2 \sigma_3 \sigma_2 \cdot \sigma_4 = \sigma_1 \cdot \sigma_3 \sigma_2 \sigma_3 \cdot \sigma_4 =$$
$$\sigma_3 \cdot \sigma_1 \sigma_2 \sigma_3 \sigma_4 = \sigma_3 \alpha$$

It follows that $\alpha \sigma_2 \alpha^{-1} = \sigma_3$, or $\sigma_3 = \alpha \sigma_2 \alpha^{-1}$.
Substituting our result for σ_2 we obtain

$$\sigma_3 = \alpha \alpha \sigma_1 \alpha^{-1} \alpha^{-1} = \alpha^2 \sigma_1 \alpha^{-2}.$$

Finally:

$$\alpha \sigma_3 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_3 = \sigma_1 \sigma_2 \sigma_4 \sigma_3 \sigma_4 = \sigma_4 \sigma_1 \sigma_2 \sigma_3 \sigma_4 = \sigma_4 \alpha$$

Consequently $\alpha \sigma_3 \alpha^{-1} = \sigma_4$. Substituting for σ_3:

$$\sigma_4 = \alpha \sigma_3 \alpha^{-1} = \alpha \alpha^2 \sigma_1 \alpha^{-2} \alpha^{-1} = \alpha^3 \sigma_1 \alpha^{-3}.$$

In one formula:

$$\sigma_i = \alpha^{i-1} \sigma_1 \alpha^{-(i-1)}.$$

Each σ_i can be expressed by α and σ_1 and therefore any braid A can be expressed by α and σ_1.

In the following we shall not make use of this result.

The formulas:

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$, if j is at least $i+2$

(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

have the following significance:

Suppose two braids A and B given by patterns. Each pattern may be used to express A and B respectively as a product of terms σ_i or σ_i^{-1}.

If $A = B$, it must in some fashion be possible to change from the expression A to the expression B. It can be shown that this can always be done by a repeated use of either formulas (1) or (2), or of simple algebraic consequences of these formulas. It is this fact one refers to if one says: the braid group has the defining relations (1) and (2). The proof is too long to be reproduced here.

We proceed now to our fundamental problem. Let us first consider a braid A in which the curves c_i connect P_i with Q_i (the Q_i with exactly the same subscript).

Suppose we remove the curve c_1. A certain braid A_1 of order $n-1$ remains. Now we reinsert a curve d_1 between P_1 and Q_1 that is not entangled at all with the other strings (this means that its projection exhibits no crossings at all). This new braid of order n we call B.

Denote now the braid AB^{-1} by C. This braid C has a peculiar property. If the first string of C is removed, then the braid that remains from the A-part of C is A_1, and A_1^{-1} is the part that remains from B^{-1}. (According to our construction, A and B differ only by their first strings.) Therefore removing the first string from C leaves $A_1 A_1^{-1} = I$ —a braid that can be combed. To be sure, C itself cannot necessarily be combed until the first string has been removed.

Suppose now that this combing operation with the last $n-1$ strings of C is performed by force in spite of the presence of the first string. Since the first string is stretchable up to any amount, it may be taken along during this combing operation. At the end the first string will be entangled in a terrible fashion, but the result will look somewhat like Figure 8. A pattern of this type is called 1-pure.

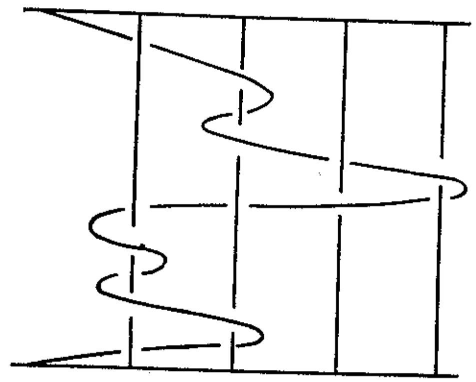

FIGURE 8

Now $AB^{-1} = C$; $AB^{-1}B = CB$; therefore $A = CB$. So A is a product of a 1-pure braid C and another braid B which is obtained from a braid of order $n-1$ by inserting a first string not meeting the others in a projection. The second string of B can be treated in the same way, and so on.

The final result is:

$$A = C_1 C_2 \ldots C_{n-1}$$

where C_i is a braid of the following kind: all strings but the i-th are vertical straight lines, and the i-th is only involved with strings of a higher number. Of course this means that for every braid A a pattern of this special kind can be found.

The solution of our fundamental problem consists in the assertion that a pattern of this type describes the braid uniquely; i.e., that in order to test whether $A = B$ for two braids whose curves c_1 connect P_1 with Q_1 one has only to bring A and B into this form and to see whether exactly the same pattern results. The proof for this fact is very involved and cannot be included here. Nor shall we describe the translation of our geometric procedure into group theoretical language.

It is clear that this procedure contains the solution of the full problem to decide whether $A = B$ for any two braids A and B given by weaving patterns. First $A = B$ means the same as $AB^{-1} = I$. The braid I connects P_1 with Q_1. Should AB^{-1} not do this, then certainly A is not equal to B. In case AB^{-1} connects each P_1 with Q_1, the previous method makes it possible to decide whether $AB^{-1} = I$ or not.

Finally let us mention an unsolved problem of the theory of braids. If we wind a braid once around an axis, close it by identifying P_1 and Q_1, and remove the lines L_1 and L_2, we obtain what we call a closed braid. Again we allow all those deformations in the course of which the curves do not cross the axis, nor each other.

The problem of classification of closed braids, at least, can be translated into a group theoretical problem. Let A and B be two open braids. The corresponding closed braids are equal if, and only if, an open braid X can be found such that

$$B = XAX^{-1}$$

A solution to this problem has not yet been found. Since in some ways closed braids resemble knots, such a solution could be applied to the problem of knots. It would also have many applications in pure mathematics.

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Ein mechanisches System mit quasiergodischen Bahnen.

Von EMIL ARTIN in Hamburg

Es sei gestattet, auf ein einfaches mechanisches System von zwei Freiheitsgraden mit quasiergodischen Bahnen hinzuweisen, zu dem der Verfasser in einem Briefwechsel mit Herrn G. Herglotz gekommen ist.

Wir wollen nämlich eine Fläche konstruieren, deren geodätische Linien „fast alle" quasiergodisch sind, also jedem Punkt beliebig nahe kommen, und zwar in jedem vorgeschriebenen Richtungsintervall.

Wir betrachten die obere Halbebene $y > 0$ der komplexen Variablen $z = x + iy$ und führen in ihr die Poincarésche Metrik ein mit dem Linienelement

(1)
$$ds = \frac{d\sigma}{y},$$

wo $d\sigma = |dz|$ das gewöhnliche euklidische Linienelement bedeutet. Die so eingeführte Metrik besitzt zwei bekannte einfache Eigenschaften.[1])

1. Sie ist invariant gegenüber allen linearen Substitutionen $z_1 = \frac{az+b}{cz+d}$ mit reellen Koeffizienten und positiver Determinante: $ad - bc > 0$. Sie ist auch invariant gegenüber der Spiegelung $z_1 = -\bar{z}$, wo $\bar{z}$ die zu z konjugiert komplexe Zahl bedeutet.

2. Die geodätischen Linien sind alle Halbkreise der oberen Halbebene, die auf der reellen Achse senkrecht aufsitzen.

Aus 1. geht die Homogeneität unserer Metrik hervor. Die Rechnung führt auf das Krümmungsmaß $K = -1$.

Nunmehr identifizieren wir alle Punkte der Halbebene, die durch eine Substitution mit ganzen Koeffizienten und der Determinante 1 zusammenhängen; wir betrachten also zwei Punkte z_1 und z_2 als „gleich", wenn:

(2)
$$z_1 = \frac{a z_2 + b}{c z_2 + d}$$

gilt, mit ganzen rationalen a, b, c, d die der Bedingung $ad - bc = 1$ genügen.

[1]) Siehe etwa W. Blaschke, Differentialgeometrie I, § 62.